JOURNAL OF APPROXIMATION THEORY 43, 90–98 (1985)

Euler $\mathscr{L}$-Splines and an Extremal Problem for Periodic Functions

H. G. TER MORSCHE AND F. SCHURER

Department of Mathematics and Computing Science, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands

Communicated by I. J. Schoenberg

Received February 8, 1984

1. INTRODUCTION AND SUMMARY

1.1. Landau's well-known inequality (cf. [5]) for twice differentiable functions may be put in the following form: if f and f'' are bounded on $\mathbb{R}$ then $\|f'\| \leqslant 2^{1/2}(\|f\| \, \|f''\|)^{1/2}$; here, and throughout this paper, $\|\cdot\|$ denotes the supremum norm. Landau's inequality is *best possible*, i.e., the constant $2^{1/2}$ cannot be replaced by a smaller one. Around 1939 Kolmogorov [4] obtained similar best possible inequalities connecting $\|f\|$, $\|f^{(n)}\|$, $\|f^{(k)}\|$ $(1 \leqslant k \leqslant n-1)$. The analogous problem for periodic functions has been dealt with by Northcott [7].

It is interesting to note that the extremal functions, i.e., the functions for which the inequalities above turn into equalities are the same for both problems; these extremal functions are the Euler splines. Cavaretta [1], who gave an elementary prqof of Kolmogorov's inequalities by first establishing them for periodic functions, showed that Euler splines also maximize the functional $\|f^{(k+1)} + \alpha f^{(k)}\|$, for any $\alpha \in \mathbb{R}$ and for $0 \leqslant k \leqslant n-2$, on the set of functions f with prescribed upper bounds for $\|f\|$ and $\|f^{(n)}\|$ $(n \geqslant 2)$.

1.2. As the main result of this paper we show that the so-called Euler $\mathscr{L}$-splines are extremal with respect to a rather general class of differential operators defined on the set of periodic functions.

Preliminary material is collected in Secion 2. Section 3 contains a proof of the main result and an example.

0021-9045/85 $3.00

2. Preliminary Notions and Results

2.1. By $W^{(n)}$ we denote the set of functions f having an absolutely continuous $(n-1)$st derivative $f^{(n-1)}$ on every compact subinterval of $\mathbb{R}$ and a (Radon–Nikodym) derivative $f^{(n)}$ that is essentially bounded on $\mathbb{R}$, i.e., $f^{(n)} \in L_\infty(\mathbb{R})$. For a given period $T > 0$ the set $W_T^{(n)}$ is then defined by

$$W_T^{(n)} = \{f \in W^{(n)} \mid f(t+T) = f(t), t \in \mathbb{R}\}.$$

Let D be the ordinary differentiation operator and let p_n be a polynomial of degree n, then the corresponding differential operator of order n is denoted by $p_n(D)$, $D^0 = I$.

Let h be a positive number and let p_n be a monic polynomial of degree n. If a function s satisfies the conditions

$$\begin{aligned} &s \in W^{(n)} \\ p_n(D)\, s(t) &= -1 \qquad (0 < t < h), \\ s(t+h) &= -s(t) \qquad (t \in \mathbb{R}), \end{aligned} \tag{2.1}$$

then s is called an *Euler $\mathscr{L}$-spline* corresponding to the operator $p_n(D)$ and with mesh distance h. It can be shown that s is uniquely determined by (2.1) if p_n has only real zeros; in this case s will be denoted by $E(p_n, h, \cdot)$.

2.2. Let p_n $(n \geqslant 2)$ be a monic polynomial of degree n having only real zeros. Furthermore, let the function $\mathscr{P}_n$ be defined by means of its Fourier series with period T, i.e., let

$$\mathscr{P}_n(t) = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} p_n^{-1}(i\omega j)\, e^{i\omega jt} \qquad (t \in \mathbb{R}), \tag{2.2}$$

where $\omega = 2\pi/T$. Then $\mathscr{P}_n \in W_T^{(n-1)}$ and (cf. ter Morsche [6, p. 137–138]) $\mathscr{P}_n$ can be written in the form

$$\mathscr{P}_n(t) = \frac{T}{2\pi i} \oint_C \frac{e^{tz}}{(1-e^{Tz})\, p_n(z)}\, dz \qquad (0 \leqslant t \leqslant T), \tag{2.3}$$

where C is a closed contour in the complex plane including the origin and the zeros of p_n, but excluding the points $z = i\omega j$ $(j = \pm 1, \pm 2, \ldots)$. It immediately follows from (2.3) that

$$p_n(D)\, \mathscr{P}_n(t) = -1 \qquad (0 < t < T). \tag{2.4}$$

Let p_k $(0 \leqslant k \leqslant n-2)$ be a monic polynomial of degree k that divides p_n. We now introduce $\mathscr{P}_{n,k}$ defined by $\mathscr{P}_{n,k} = p_k(D)\,\mathscr{P}_n$; on account of (2.2), $\mathscr{P}_{n,k}$ corresponds to $p_{n,k} := p_k^{-1} p_n$ in the same way as $\mathscr{P}_n$ corresponds to p_n in (2.2).

A representation formula for the elements of the set $W_T^{(n)}$ is given in the following lemma.

LEMMA 2.1. *If $f \in W_T^{(n)}$ then*

$$f(t) = T^{-1} \int_0^T f(\tau)\, d\tau + T^{-1} \int_0^T \mathscr{P}_n(t-\tau)\, p_n(D) f(\tau)\, d\tau \qquad (t \in \mathbb{R}). \tag{2.5}$$

For a proof of this lemma the reader is referred to Golomb [3] or ter Morsche [6, Lemma 6.3.1].

2.3. In Section 3.1 we need an estimate on the number of zeros of various derivatives of $\mathscr{P}_n$ in the interval $(0, T]$. The following lemma is used for that purpose. Here $\operatorname{Ker}(p_n)$ denotes the kernel of $p_n(D)$, i.e., the set of real-valued functions f for which $p_n(D) f(t) = 0$ $(t \in \mathbb{R})$. By $Z_f(J)$ we denote the number of zeros of f in the set J, counting multiplicities.

LEMMA 2.2. *Let p_n be a monic polynomial of degree n having only real zeros, and let r be a nonnegative integer. Furthermore, let $f \not\equiv 0$ have the properties*

$$\begin{aligned} &\text{(i)} && f \in \operatorname{Ker}(p_n), \\ &\text{(ii)} && f^{(j)}(0) = f^{(j)}(T) \qquad (j = 0, 1, \ldots, n-r-1). \end{aligned} \tag{2.6}$$

Then

$$\begin{aligned} Z_f((0, T]) &\leqslant r - 1 \qquad (r \text{ odd}), \\ &\leqslant r \qquad\quad\;\; (r \text{ even}). \end{aligned} \tag{2.7}$$

Proof. We distinguish between the cases $r \geqslant n$ and $0 \leqslant r < n$. If $r \geqslant n$ then condition (ii) of (2.6) is void. Since p_n only has real zeros, a nontrivial function $f \in \operatorname{Ker}(p_n)$ has at most $n-1$ zeros in $\mathbb{R}$, and inequality (2.7) obviously holds. Now let $0 \leqslant r < n$, and let $q \not\equiv 0$ be a continuously differentiable function satisfying $q(0) = q(T)$, $q'(0) = q'(T)$. Then for any $\lambda \in \mathbb{R}$,

$$Z_q((0, T]) \leqslant Z_{q' - \lambda q}((0, T]). \tag{2.8}$$

This inequality may be verified by writing

$$q'(t) - \lambda q(t) = e^{\lambda t} \frac{d}{dt} \left(e^{-\lambda t} q(t)\right)$$

and using Rolle's theorem. We note that $Z_q((0, T])$ is even if $q(0) \neq 0$. Denoting the zeros of p_n by $\alpha_1, \alpha_2, ..., \alpha_n$, we introduce the polynomials p_{r+1} and p_{n-r-1} defined by

$$p_{r+1}(x) = (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_{r+1}),$$
$$p_{n-r-1}(x) = (x-\alpha_{r+2})(x-\alpha_{r+3}) \cdots (x-\alpha_n).$$

Then $g := p_{n-r-1}(D)f \in \operatorname{Ker}(p_{r+1})$ and in view of (ii) of (2.6) we conclude that $g(0) = g(T)$. We proceed by first assuming that $g \not\equiv 0$. As p_{r+1} has only real zeros it follows that $Z_g(J) \leqslant r$ for any set $J \subset \mathbb{R}$. Since

$$g(t) = e^{\alpha_{r+2}t} \frac{d}{dt} e^{-\alpha_{r+2}t} e^{\alpha_{r+3}t} \frac{d}{dt} e^{-\alpha_{r+3}t} \cdots e^{\alpha_n t} \frac{d}{dt} e^{-\alpha_n t} f(t),$$

repeated application of (2.8) yields

$$Z_f((0, T]) \leqslant Z_g((0, T]).$$

Hence, $Z_f((0, T]) \leqslant r$. If $Z_f((0, T]) = r$ then obviously $Z_g((0, T]) = r$. It follows that $g(0) \neq 0$ and therefore that r is even, since otherwise one would have $Z_g((0, T]) > r$. This proves (2.7) in case $g \not\equiv 0$. It remains to consider $g \equiv 0$. Then $f \in \operatorname{Ker}(p_{n-r-1})$ and in view of (ii) of (2.6), f is periodic. If $p_{n-r-1}(0) \neq 0$ then $f \equiv 0$, contradicting the hypotheses of the lemma; however, if $p_{n-r-1}(0) = 0$ then f is a nonzero constant function for which (2.7) clearly holds. This proves the lemma. ∎

2.4. In order to formulate the next lemma we need the following definition.

DEFINITION 2.3. $U = \{u \in L_\infty([0, T]) \mid \|u\| \leqslant 1, \int_0^T u(\tau)\, d\tau = 0\}$.

LEMMA 2.4. *Let g be an arbitrary real-valued nonconstant analytic function defined on $[0, T]$. Then there is a unique determined real constant c_0 such that*

$$\max_{u \in U} \int_0^T g(\tau)\, u(\tau)\, d\tau = \int_0^T |g(\tau) - c_0|\, d\tau. \tag{2.9}$$

Moreover, functions $u \in U$ for which this maximum is attained are given by

$$u(t) = \operatorname{sgn}(g(t) - c_0) \qquad (a.e. \text{ on } [0, T]). \tag{2.10}$$

Proof. For every $u \in U$ and $c \in \mathbb{R}$ one has

$$\int_0^T g(\tau)\, u(\tau)\, d\tau = \int_0^T (g(\tau) - c)\, u(\tau)\, d\tau \leqslant \int_0^T |g(\tau) - c|\, d\tau.$$

Hence

$$\int_0^T g(\tau)\,u(\tau)\,d\tau \leqslant \min_{c\in\mathbb{R}} \int_0^T |g(\tau)-c|\,d\tau.$$

So the L_1-distance of g to the set of constant functions has to be determined. Since by assumption g is a real-valued nonconstant analytic function, it coincides with any constant c in at most finitely many points of $[0, T]$. According to a well-known characterization theorem for L_1-approximation (cf. Cheney [2, p. 220]), the best approximation c_0 to g is uniquely determined by

$$\int_0^T \operatorname{sgn}(g(\tau)-c_0)\,d\tau = 0. \tag{2.11}$$

Formula (2.9) now immediately follows by taking $u(t)=\operatorname{sgn}(g(t)-c_0)$. With respect to the second assertion of the lemma we note that for functions $u \in U$ the equality

$$\int_0^T (g(\tau)-c_0)\,u(\tau)\,d\tau = \int_0^T |g(\tau)-c_0|\,d\tau$$

holds if and only if u is given by (2.10). ∎

3. An Extremal Property of Euler $\mathscr{L}$-Splines

3.1. Our main result is the following theorem.

THEOREM 3.1. *Let p_n $(n \geqslant 2)$ be a monic polynomial of degree n having only real zeros with $p_n(0)=0$. Furthermore, let p_k $(0 \leqslant k \leqslant n-2)$ be a monic polynomial of degree k that divides p_n. Then the following two inequalities hold:*

(i) *if $p_k(0)=0$ then for all $\alpha \in \mathbb{R}$ and all $f \in W_T^{(n)}$,*

$$\|p_k(D)(D+\alpha I)f\| \leqslant p_k(D)(D+\alpha I)\,E(p_n, T/2, \cdot)\|\,\|p_n(D)f\|; \tag{3.1}$$

(ii) *if $p_k(0) \neq 0$ then for all $f \in W_T^{(n)}$,*

$$\|p_k(D)\,Df\| \leqslant \|p_k(D)\,DE(p_n, T/2, \cdot)\|\,\|p_n(D)f\|. \tag{3.2}$$

Moreover, equality in (3.1) *or* (3.2) *holds if and only if $\beta \in \mathbb{R}$ and $\xi \in (0, T]$ exist such that*

$$f(t) = \beta E(p_n, T/2, t-\xi) \qquad (t \in \mathbb{R}).$$

Proof. Without loss of generality we may assume that $\|p_n(D)f\| \leqslant 1$. Accordingly, define

$$\bar{W}_T^{(n)} = \{f \in W_T^{(n)} \mid \|p_n(D)f\| \leqslant 1\}.$$

In order to prove (3.1) and (3.2) one has to determine

$$\sup_{f \in \bar{W}_T^{(n)}} \|p_k(D)(D + \alpha I)f\|, \tag{3.3}$$

with $\alpha = 0$ in case $p_k(0) \neq 0$. As the set $W_T^{(n)}$ is invariant under translation of arguments, (3.3) equals

$$\sup_{f \in \bar{W}_T^{(n)}} |p_k(D)(D + \alpha I)f(T)|. \tag{3.4}$$

Applying $p_k(D)(D + \alpha I)$ to (2.5) and putting $t = T$, for any $f \in \bar{W}_T^{(n)}$ we obtain the relation

$$p_k(D)(D + \alpha I)f(T) = T^{-1} \int_0^T G(T - \tau)\, p_n(D)f(\tau)\, d\tau, \tag{3.5}$$

where G is given by (cf. **2.2**)

$$G(t) = p_k(D)(D + \alpha I)\, \mathscr{S}_n(t) = (D + \alpha I)\, \mathscr{S}_{n,k}(t). \tag{3.6}$$

Since $p_n(0) = 0$ one has $\int_0^T p_n(D)f(\tau)\, d\tau = 0$; this, together with $\|p_n(D)f\| \leqslant 1$, implies that $p_n(D)f \in U$. By (3.5) and on account of Definition 2.3 it follows that

$$\sup_{f \in \bar{W}_T^{(n)}} \|p_k(D)(D + \alpha I)f\| = \max_{u \in U} T^{-1} \int_0^T G(T - \tau)\, u(\tau)\, d\tau. \tag{3.7}$$

Because of (2.4), G satisfies the differential equation

$$p_{n,k}(D)\, G(t) = -\alpha \qquad (0 < t < T),$$

and thus coincides with an analytic function on $(0, T)$. Moreover, G is not constant since (cf. (2.2)) otherwise $(i\omega j + \alpha)\, p_k(i\omega j)$ would be zero for all $j = 0, \pm 1, \pm 2, \ldots$, which cannot occur since by assumption $p_k \not\equiv 0$. Consequently, we may apply Lemma 2.4 to (3.7). This yields a constant c_0 uniquely determined by (cf. (2.11)),

$$\int_0^T \operatorname{sgn}(G(T - \tau) - c_0)\, d\tau = 0.$$

Let $H(t) := G(t) - c_0$, then H satisfies the differential equation

$$Dp_{n,k}(D)\,H(t) = 0 \qquad (0 < t < T).$$

Moreover, $H^{(j)}(0) = H^{(j)}(T)$ $(j = 0, 1, \ldots, n-k-3)$. In view of Lemma 2.2 one has $Z_H((0, T]) \leqslant 2$. Since $\int_0^T \operatorname{sgn} H(\tau)\,d\tau = 0$ it follows that either H has precisely one zero in $(0, T)$ located at $T/2$, or H has precisely two zeros in $(0, T)$ a distance $T/2$ apart. In any case H has equidistant zeros in $\mathbb{R}$ with distance $T/2$. These observations ascertain that a function $f \in \tilde{W}_T^{(n)}$ yielding the supremum in (3.4) has the property $p_n(D)f(t) = \operatorname{sgn}(H(T-t))$. Moreover, any function yielding the supremum in (3.3) satisfies the differential equation

$$p_n(D)f(t) = \operatorname{sgn}(H(\eta - t)) \qquad (t \in \mathbb{R}),$$

for some $\eta \in (0, T]$. Taking into account the definition of the Euler $\mathscr{L}$-splines (cf. **2.1**), we conclude that an extremal function f has the form

$$f(t) = \beta E(p_n, T/2, t - \xi)$$

for some $\beta \in \mathbb{R}$ and some $\xi \in (0, T]$, i.e., it is an appropriate multiple of an Euler $\mathscr{L}$-spline. This completes the proof of Theorem 3.1. ∎

Remark. If in case (ii) we take in particular $p_n(D) = D^n$ and $k = 0$, then (3.2) implies Northcott's theorem. We further note that results similar to Theorem 3.1 have been derived by Golomb [3] for specific subsets of $W_T^{(n)}$ and for specific functionals.

3.2. As an application of Theorem 3.1 we consider the following example.

EXAMPLE. Given $n \in \mathbb{N}$ and $\gamma > 0$ let

$$p_{2n+1}(D) = D(D^2 - \gamma^2 I)(D^2 - (2\gamma)^2 I) \cdots (D^2 - (n\gamma)^2 I). \tag{3.8}$$

According to (3.1) one has, taking $p_k(D) = D$ and $\alpha = 0$,

$$\|f''\| \leqslant \|E''(p_{2n+1}, T/2, \cdot)\|\,\|p_{2n+1}(D)f\| \qquad (f \in W_T^{(2n+1)}).$$

Applying formula 3.2.30 in ter Morsche [6, p. 67], we obtain by elementary calculations

$$E(p_{2n+1}, T/2, t) = \frac{(-1)^{n+1}}{(n!)^2\,\gamma^{2n}}(t - T/4) - \frac{2}{\gamma^{2n+1}} \sum_{k=1}^{n} \frac{(-1)^{n-k}\sinh((t - T/4)\,k\gamma)}{(n+k)!\,(n-k)!\cosh(k\gamma T/4)}, \tag{3.9}$$

where $0 \leqslant t \leqslant T/2$.

A careful count of the zeros of $E'''(p_{2n+1}, T/2, \cdot)$ shows that on $[0, T/2]$ this derivative only vanishes at the endpoints of $[0, T/2]$. So $|E''(p_{2n+1}, T/2, \cdot)|$ attains its maximum at $t=0$, and using (3.9) we get

$$\begin{aligned}\|E''(p_{2n+1}, T/2, \cdot)\| &= \frac{2}{\gamma^{2n-1}} \left| \sum_{k=1}^{\infty} \frac{(-1)^{n-k}\, k \tanh(k\gamma T/4)}{(n+k)!\,(n-k)!} \right| \\ &= \frac{1}{(2n)!\,\gamma^{2n-1}} \left| \sum_{k=0}^{2n} (-1)^k (n-k) \binom{2n}{k} \tanh((n-k)\gamma T/4) \right|. \end{aligned} \tag{3.10}$$

As is apparent from (3.8) the polynomial case $p_{2n+1}(D) = D^{2n+1}$ is obtained by letting $\gamma \downarrow 0$. In order to evaluate (3.10) for $\gamma \downarrow 0$ we use the identities

$$\sum_{k=0}^{2n} (-1)^k (n-k)^{2j} \binom{2n}{k} = (2n)!\,\delta_{j,n} \qquad (j = 0, 1, 2, ..., n), \tag{3.11}$$

which are easily verified.

For small x let $\tanh x = \sum_{j=1}^{\infty} c_j x^{2j-1}$. Then for sufficiently small γ

$$\begin{aligned}&\sum_{k=0}^{2n} (-1)^k (n-k) \binom{2n}{k} \tanh((n-k)\gamma T/4) \\ &\qquad = \sum_{j=1}^{\infty} c_j \left(\frac{T}{4}\right)^{2j-1} \gamma^{2j-1} \sum_{k=0}^{2n} (-1)^k (n-k)^{2j} \binom{2n}{k}.\end{aligned}$$

In view of (3.10) and (3.11) we conclude that

$$\lim_{\gamma \downarrow 0} \|E''(p_{2n+1}, T/2, \cdot)\| = |c_n|\,(T/4)^{2n-1}.$$

By the residue theorem

$$c_n = \frac{1}{2\pi i} \oint_C \frac{\tanh(z)}{z^{2n}}\, dz,$$

C being a closed contour including $z=0$, but excluding the poles of $\tanh(z)$. Since the sum of all residues of $\tanh(z)/z^{2n}$ is zero, it follows that

$$c_n = \frac{-2}{\pi^{2n}} \sum_{j=0}^{\infty} \left(j + \frac{1}{2}\right)^{-2n}.$$

Consequently,

$$\lim_{\gamma \downarrow 0} \|E''(p_{2n+1}, T/2, \cdot)\| = \frac{8}{T} (T/2\pi)^{2n} \sum_{j=0}^{\infty} (2j+1)^{-2n}.$$

Taking $T=2\pi$ we obtain

$$\|f''\| \leqslant \frac{4}{\pi} \|f^{(2n+1)}\| \sum_{j=0}^{\infty} (2j+1)^{-2n} \qquad (f \in W_{2\pi}^{(2n+1)}),$$

which agrees with Northcott's theorem.

Acknowledgments

The authors are indebted to F. W. Steutel for a number of useful comments and a critical reading of the manuscript.

References

1. A. S. Cavaretta, An elementary proof of Kolmogorov's theorem, *Amer. Math. Monthly* **81** (1974), 480–486.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. Golomb, M. "Some Extremal Problems for Differentiable Periodic Functions in $L_\infty(\mathbb{R})$," MRC Technical Summary Report No. 1069, University of Wisconsin, Madison, 1970.
4. A. N. Kolmogorov, On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval, *Uchen. Zap. Moskov. Gos. Univ. Mat.* **30** (1939), 3–13; *Amer. Math. Soc. Transl.* **2** (1962), 233–243.
5. E. Landau, Einige Ungleichungen für zweimal differentierbare Funktionen, *Proc. London Math. Soc.* **13** (1913), 43–49.
6. H. G. ter Morsche, "Interpolational and Extremal Properties of $\mathscr{L}$-Spline Functions," Thesis, Eindhoven University of Technology, Eindhoven, 1982.
7. D. G. Northcott, Some inequalities between periodic functions and their derivatives, *J. London Math. Soc.* **14** (1939), 198–202.